



Diffusion in Tube Dialyzer

Yohannes Nigatie

Department of Mathematics, College of Natural and Computational Sciences,
University of Gondar, Gondar, Ethiopia.

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ABSTRACT: Nowadays, kidney failure is a problem of many peoples in the world. We know that the main function of kidney is maintaining the chemical quality of blood particularly removing urea through urine. But when they malfunction, the pathologic state known as uremia results in a condition in which the urea is retained in the body. Failure of the kidney results in building up of harmful wastes and excess fluids in the body. Kidney diseases (failures) can be due to infections, high blood pressure (hypertension), diabetes, and/or extensive use of medication. The best form of treatment is the implantation of a healthy kidney from a donor. However, this is often not possible due to the limited availability of human organs. Chronic kidney failure requires the treatment using a tube dialyzer called dialysis. Blood is taken out of the body and passes through a special membrane that removes waste and extra fluids. The clean blood is then returned to the body. The process is controlled by a dialysis machine (tube dialyzer) which is equipped with a blood pump and monitoring systems to ensure safety. So this article investigates the real application of mathematics (diffusion) in medical science, and it also contains the mathematical formulation and interpretation of tube dialyzer in relation to diffusion.

KEYWORDS: Tube dialyzer, diffusion, uremia, dialysate

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CORRESPONDING AUTHOR: Yohannes Nigatie, Department of Mathematics, College of Natural and Computational Sciences, University of Gondar, P. O. Box 196, Gondar, Ethiopia. Email: abynz09@gmail.com

Introduction

Kidneys are two bean-shaped organs which are situated in the rear part of the abdominal cavity of our body. They perform several important functions within the human body, such as cleaning the blood and waste products of the body, getting rid of the extra fluids in the body through the urine, and helping in the production of red blood cells. The main function of kidney is to maintain the chemical quality of blood by removing the waste products contributed to the blood stream by the metabolic process in the human body. In particular, kidneys help to remove urea through urine. When they malfunction, the pathologic state known as uremia results in a condition in which the urea is retained in the body. It leads to many severe complications which may prove fatal. When uremia cannot be cured by medicine, the only alternative is to take the impure blood out of the body, remove urea from it, and then return the purified blood to the body. The engineering device used for this purpose is called the tube dialyzer which serves almost as the same purpose as the human kidney.

Fluids

A fluid is defined as a substance that can flow, or a substance that does not maintain a fixed shape. Gases and liquids are fluids. That is, the particles making up the substances continuously change their positions relative to one another. Fluids do not offer any lasting resistance to the displacement of one layer over another when a shear force is applied.

Diffusion

Diffusion is a random movement of molecules, which results in molecules moving from an area of higher concentration to an area of lower concentration.

Fluid flow through tube dialyzer

Failure of the kidney results in building up of harmful wastes and excess fluids in the body. Kidney diseases (failures) can be due to infections, high blood pressure (hypertension), diabetes, and/or extensive use of medication. The best form of treatment is the implantation of a healthy kidney from a donor. However, this is often not possible due to the limited availability of human organs. Chronic kidney failure requires the treatment using a tube dialyzer called dialysis. Blood is taken out of the body and passes through a special membrane that removes waste and extra fluids. The clean blood is then returned to the body. The process is controlled by a dialysis machine which is equipped with a blood pump and monitoring systems to ensure safety.

Uremia

Uremia is a condition in which the urea concentration in blood is chronically elevated, reflecting an inability to remove from the body the end products of protein metabolism.

Dialysate

Dialysate is a buffered electrolyte solution, usually containing glucose at or above physiologic concentration, circulated



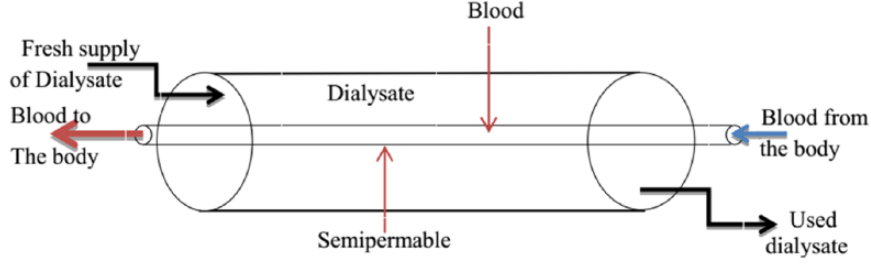


Figure 1. Circular duct hemodialyzer.¹

through the water compartment of a hemodialyzer to control diffusional transport of small molecules across the membranes and achieve the blood concentrations desired.

Tube dialyzer

Tube dialyzer is the piece of equipment that actually filters the blood. One of the most popular types is the hollow-fiber dialyzer.

Hollow-fiber dialyzers are life-saving devices that extend life by removing toxins that accumulate in patients with end-stage kidney disease. Small solutes are normally removed by diffusion, whereas larger solutes are mostly removed by convection through ultrafiltration. Therefore, the efficiency of a hollow-fiber dialyzer depends on its ability to facilitate both diffusion and convection processes.

From a mathematical modeling perspective, typical hollow-fiber dialyzer geometry uses countercurrent blood and dialysate flows separated by hollow-fiber membrane. Blood flows inside the hollow fibers (tubes), whereas dialysate flows outside the hollow-fiber region counter currently.

In hollow-fiber dialyzer, the blood is run through a bundle of very thin capillary-like tubes, and the dialysate is pumped in a chamber of bathing fibers (Figure 1)¹.

A larger dialyzer will usually translate to an increased membrane area and thus there is an increase in the amount of solutes removed from the patient's blood. Different types of dialyzers have different clearances for different solutes.

Basic Equations

Diffusion equation

The basic partial differential equation that governs diffusion in tube dialyzer is the diffusion equation with the convective term²:

$$D \left(\frac{\partial^2 c}{\partial r^2} + \frac{1}{r} \frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial z^2} \right) = v \frac{\partial c}{\partial z} = v_m \left(1 - \frac{r^2}{R^2} \right) \frac{\partial c}{\partial z} \quad (1)$$

where $c(r, z)$ is the concentration of urea in the blood at the point (r, z) , $v(r)$ is the velocity in the fully developed flow at this point, v_m is the maximum velocity, R is the radius of the duct, and D is the diffusion coefficient. We are assuming steady-state laminar Newtonian fluid flow constant physical

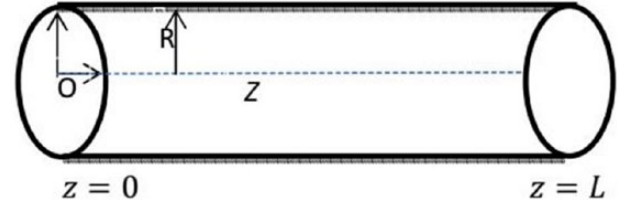


Figure 2. Circular duct for blood flow.

properties (including constant permeability), a straight duct without any sagging or osmotic or ultrafiltration effect and fully developed blood flow (Figure 2).

The magnitude of the convective term as compared with the magnitude of the longitudinal diffusion term is given by the dimensionless Peclet number:

$$P_e = \frac{v_m R}{D} \quad (2)$$

Which may be as large as 15 000 for the hemodialyzer so that the longitudinal diffusion term can be neglected and equation (1) is simplified as follows:

$$D \left(\frac{\partial^2 c}{\partial r^2} + \frac{1}{r} \frac{\partial c}{\partial r} \right) = v_m \left(1 - \frac{r^2}{R^2} \right) \frac{\partial c}{\partial z} \quad (3)$$

The boundary conditions for solving equation (3) are as follows:

1. Flow symmetry about the axis and finite concentration at $r=0$, which give the following:

$$\frac{\partial c}{\partial r} = 0 \text{ at } r = 0 \quad (4)$$

2. Assumption of constant entry concentration gives the following:

$$c = c_{in} \text{ at } z = 0, \quad 0 \leq r \leq R \quad (5)$$

3. Assumptions of permeability p and constant c_d in the dialysate yield the following:

$$-D \frac{\partial c}{\partial r} = p(c - c_d) \text{ at } r = R, \quad z > 0 \quad (6)$$

Introducing the nondimensional quantities,

$$\bar{c} = \frac{c - c_d}{c_{in} - c_d}, \bar{r} = \frac{r}{R}, \bar{z} = \frac{z}{Rp_\epsilon}, p_\epsilon = \frac{v_m R}{D} \quad (7)$$

Therefore, using the above boundary conditions, the solution of equation (3) will be as follows.

Now, let us differentiate \bar{c} with respect to r twice, ie,

$$\begin{aligned} \frac{\partial \bar{c}}{\partial r} &= \frac{\partial}{\partial r} \left(\frac{c - c_d}{c_{in} - c_d} \right) = \frac{\partial}{\partial r} \left(\frac{c}{c_{in} - c_d} \right) - \frac{\partial}{\partial r} \left(\frac{c_d}{c_{in} - c_d} \right) \\ &\Rightarrow \frac{\partial \bar{c}}{\partial r} = \frac{1}{c_{in} - c_d} \frac{\partial c}{\partial r} \end{aligned} \quad (8)$$

Then,

$$\begin{aligned} \frac{\partial^2 \bar{c}}{\partial r^2} &= \frac{\partial}{\partial r} \left(\frac{\partial \bar{c}}{\partial r} \right) = \frac{\partial}{\partial r} \left(\frac{1}{c_{in} - c_d} \frac{\partial c}{\partial r} \right) = \frac{1}{c_{in} - c_d} \frac{\partial^2 c}{\partial r^2} \\ &\Rightarrow \frac{\partial^2 \bar{c}}{\partial r^2} = \frac{1}{c_{in} - c_d} \frac{\partial^2 c}{\partial r^2} \end{aligned} \quad (9)$$

But from equation (7), we have

$$r = R\bar{r} \Rightarrow \partial r = R\partial\bar{r} \text{ and } r^2 = (R\bar{r})^2 \Rightarrow \partial r^2 = R^2\partial\bar{r}^2 \quad (10)$$

Then, substituting the values of equation (10) in equations (8) and (9), we will obtain the following results:

$$\begin{aligned} \frac{\partial \bar{c}}{R\partial\bar{r}} &= \frac{1}{c_{in} - c_d} \frac{\partial c}{\partial r} \\ &\Rightarrow \frac{\partial \bar{c}}{\partial r} = \frac{c_{in} - c_d}{R} \frac{\partial \bar{c}}{\partial \bar{r}} \end{aligned} \quad (11)$$

$$\begin{aligned} \frac{\partial^2 \bar{c}}{\partial r^2} &= \frac{1}{c_{in} - c_d} \frac{\partial^2 c}{R^2\partial\bar{r}^2} \\ &\Rightarrow \frac{\partial^2 \bar{c}}{R^2\partial\bar{r}^2} = \frac{1}{c_{in} - c_d} \frac{\partial^2 c}{\partial r^2} \\ &\Rightarrow \frac{\partial^2 c}{\partial r^2} = \frac{c_{in} - c_d}{R^2} \frac{\partial^2 \bar{c}}{\partial \bar{r}^2} \end{aligned} \quad (12)$$

Again, let us differentiate \bar{c} with respect to z

$$\begin{aligned} \frac{\partial \bar{c}}{\partial z} &= \frac{\partial}{\partial z} \left(\frac{c - c_d}{c_{in} - c_d} \right) = \frac{\partial}{\partial z} \left(\frac{c}{c_{in} - c_d} \right) - \frac{\partial}{\partial z} \left(\frac{c_d}{c_{in} - c_d} \right) \\ &\Rightarrow \frac{\partial \bar{c}}{\partial z} = \frac{1}{c_{in} - c_d} \frac{\partial c}{\partial z} \end{aligned} \quad (13)$$

Again, from equation (7) we have $\bar{z} = z / Rp_\epsilon \Rightarrow z = Rp_\epsilon \bar{z} \Rightarrow \partial z = Rp_\epsilon \partial \bar{z}$

Then, substituting this in equation (12) gives the following:

$$\frac{\partial \bar{c}}{Rp_\epsilon \partial \bar{z}} = \frac{1}{c_{in} - c_d} \frac{\partial c}{\partial z}, \text{ but } p_\epsilon = \frac{Rv_m}{D}$$

Again using substitution and by multiplying both the sides of the above equation by $c_{in} - c_d$ give the following:

$$\frac{\partial c}{\partial z} = \frac{D(c_{in} - c_d)}{R^2 v_m} \frac{\partial \bar{c}}{\partial \bar{z}} \quad (14)$$

Now, substitute equations (14), (16), and (18) in equation (17) as follows:

$$\begin{aligned} D \left(\frac{\partial^2 c}{\partial r^2} + \frac{1}{r} \frac{\partial c}{\partial r} \right) &= v_m \left(1 - \frac{r^2}{R^2} \right) \frac{\partial c}{\partial z} \\ &\Rightarrow D \left(\frac{c_{in} - c_d}{R^2} \frac{\partial^2 \bar{c}}{\partial \bar{r}^2} + \frac{1}{R\bar{r}} \frac{c_{in} - c_d}{R} \frac{\partial \bar{c}}{\partial \bar{r}} \right) \\ &= v_m (1 - \bar{r}^2) \frac{D(c_{in} - c_d)}{R^2 v_m} \frac{\partial \bar{c}}{\partial \bar{z}} \end{aligned}$$

Then, simplifying the above equation yields the following:

$$\frac{\partial^2 \bar{c}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{c}}{\partial \bar{r}} = (1 - \bar{r}^2) \frac{\partial \bar{c}}{\partial \bar{z}} \quad (15)$$

Using equation (4), ie, at $r = 0$, $\partial c / \partial r = 0$

$$\begin{aligned} \frac{\partial c}{\partial r} &= \frac{c_{in} - c_d}{R} \frac{\partial \bar{c}}{\partial \bar{r}} = 0 \\ &\Rightarrow \frac{c_{in} - c_d}{R} \frac{\partial \bar{c}}{\partial \bar{r}} = 0 \\ &\Rightarrow \frac{\partial \bar{c}}{\partial \bar{r}} = 0 \end{aligned} \quad (16)$$

Then, using equation (5), ie, $z = 0$, $c = c_{in}$

$$\begin{aligned} \bar{c} &= \frac{c - c_d}{c_{in} - c_d} \Rightarrow c = \bar{c}(c_{in} - c_d) + c_d \\ &\Rightarrow c = \bar{c}(c_{in} - c_d) + c_d = c_{in} \\ &\Rightarrow \bar{c}(c_{in} - c_d) = c_{in} - c_d \text{ at } \bar{z} = 0, \quad 0 \leq \bar{r} \leq 1 \\ &\Rightarrow \bar{c} = 1 \text{ at } \bar{z} = 0, \quad 0 \leq \bar{r} \leq 1 \end{aligned} \quad (17)$$

From equation (6), at $r = R$ we have

$$\begin{aligned} -D \frac{\partial c}{\partial r} &= p(c - c_d) \\ &\Rightarrow -D \frac{\partial c}{\partial r} = -D \frac{c_{in} - c_d}{R} \frac{\partial \bar{c}}{\partial \bar{r}} \\ &= P(c - c_d) \text{ at } R\bar{r} = R, \quad z = Rp_\epsilon \bar{z} > 0 \\ &\Rightarrow \frac{\partial \bar{c}}{\partial \bar{r}} + \frac{PR}{D} \frac{c - c_d}{c_{in} - c_d} = 0 \text{ at } \bar{r} = 1, \quad \bar{z} > 0 \end{aligned}$$

Generally, the system of equations from equations (3) to (6) gives the following:

$$\frac{\partial^2 \bar{c}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{c}}{\partial \bar{r}} = (1 - \bar{r}^2) \frac{\partial \bar{c}}{\partial \bar{z}} \quad (18)$$

$$\bar{c} = 1 \text{ at } \bar{z} = 0, \quad 0 \leq \bar{r} \leq 1 \quad (19)$$

$$\frac{\partial \bar{c}}{\partial \bar{z}} = 0 \text{ at } \bar{r} = 0, \bar{z} > 0 \quad (20)$$

$$\frac{\partial \bar{c}}{\partial \bar{z}} + Sh_w \bar{c} = 0 \text{ at } \bar{r} = 1, \bar{z} > 0 \quad (21)$$

where

$$Sh_w = \frac{pR}{D} \quad (22)$$

is called the Sherwood wall number because it is obtained by multiplying the mass transfer coefficient in the usual Sherwood number by the wall permeability.

To solve equation (18) subject to equations (19) to (21), we have an alternative method, namely, separation of variables methods which will be discussed thoroughly in the following section.

Diffusion in Tube Dialyzer

The main objective of this article is to study blood flow through tube dialyzer.

The basic partial differential equation that governs diffusion in tube dialyzer is the diffusion equation.

Method of separation of variables for solving diffusion equation

From the above section, we know that the diffusion equation in its simplified form is as follows:

$$\frac{\partial^2 \bar{c}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{c}}{\partial \bar{r}} = (1 - \bar{r}^2) \frac{\partial \bar{c}}{\partial \bar{z}} \quad (23)$$

where $\bar{c} = c - c_d / c_{in} - c_d$, $\bar{r} = r / R$ and $\bar{z} = z / Rp_e$.

Therefore, to solve the above equation using the method of separation of variables, let us start by assuming that the solution is of the following form¹:

$$\bar{c} = \sum_{n=0}^{\infty} A_n R_n(\bar{r}) e^{(-\lambda_n^2 \bar{z})} \quad (24)$$

where λ_n 's are the eigenvalues and R_n 's are the corresponding eigenfunctions, and A_n 's are to be determined from the condition of orthogonality of R_n 's.

Substituting equation (24) in equations (18) to (22), we obtain the following:

$$\frac{d^2 R_n}{d\bar{r}^2} + \frac{1}{\bar{r}} \frac{dR_n}{d\bar{r}} + \lambda_n^2 (1 - \bar{r}^2) R_n = 0 \quad (25)$$

$$\sum_{n=1}^{\infty} A_n R_n(\bar{r}) = 1 \quad (26)$$

$$\frac{dR_n}{d\bar{r}} = 0 \text{ at } \bar{r} = 0 \quad (27)$$

$$\frac{dR_n}{d\bar{r}} + Sh_w R_n = 0 \text{ at } \bar{r} = 1 \quad (28)$$

Assuming Frobenius infinite series solution to equation (25), ie,

$$R_n(\bar{r}) = \sum_{m=0,2,4,\dots} N_{m,n} \bar{r}^m \quad (29)$$

Now, substitute equation (29) in equation (27) at $\bar{r} = 0$:

$$\begin{aligned} \frac{dR_n}{d\bar{r}} &= \sum_{m=0}^{\infty} m N_{m,n} \bar{r}^{m-1} \\ \Rightarrow \sum_{m=0}^{\infty} m N_{m,n} (0)^{m-1} &= 0, \text{ at } \bar{r} = 0 \end{aligned} \quad (30)$$

Again, to obtain suitable values that can be substitute in equation (25), let us differentiate equation (29) twice with respect to \bar{r} , ie,

$$\frac{d^2 R_n(\bar{r})}{d\bar{r}^2} = \sum_{m=0,2,4,\dots} m(m-1) N_{m,n} \bar{r}^{m-2} \quad (31)$$

$$\Rightarrow \frac{d^2 R_n(\bar{r})}{d\bar{r}^2} = \sum_{m=0,2,4,\dots} m(m-1) N_{m,n} \bar{r}^{m-2} \quad (32)$$

Then, substituting equations (29), (31), and (32) in equation (25), we will obtain the following:

$$\begin{aligned} \sum_{m=0,2,\dots}^{\infty} m(m-1) N_{m,n} \bar{r}^{m-2} + \frac{1}{\bar{r}} \sum_{m=0,2,\dots}^{\infty} m N_{m,n} \bar{r}^{m-1} + \\ \lambda_n^2 (1 - \bar{r}^2) \sum_{m=0,2,\dots}^{\infty} N_{m,n} \bar{r}^m = 0 \end{aligned} \quad (33)$$

And multiplying both the sides of equation (33), by \bar{r}^2 , we get the following:

$$\begin{aligned} \sum_{m=0}^{\infty} m(m-1) N_{m,n} \bar{r}^m + \sum_{m=0}^{\infty} m N_{m,n} \bar{r}^m + \\ \lambda_n^2 \left(\sum_{m=0}^{\infty} N_{m,n} \bar{r}^{m+2} - \sum_{m=0}^{\infty} N_{m,n} \bar{r}^{m+4} \right) = 0 \end{aligned} \quad (34)$$

$$\begin{aligned} \Rightarrow \sum_{m=0}^{\infty} N_{m,n} \bar{r}^m (m(m-1) + m) + \lambda_n^2 \\ \left(\sum_{m=0}^{\infty} N_{m,n} \bar{r}^{m+2} - \sum_{m=0}^{\infty} N_{m,n} \bar{r}^{m+4} \right) = 0 \end{aligned} \quad (35)$$

$$\begin{aligned} \Rightarrow \sum_{m=0}^{\infty} m^2 N_{m,n} \bar{r}^m + \lambda_n^2 \left(\sum_{m=0}^{\infty} N_{m,n} \bar{r}^{m+2} - \sum_{m=0}^{\infty} N_{m,n} \bar{r}^{m+4} \right) = 0 \\ \Rightarrow \sum_{m=0}^{\infty} m^2 N_{m,n} \bar{r}^m + \lambda_n^2 \left(\sum_{m=2}^{\infty} N_{m-2,n} \bar{r}^m - \sum_{m=4}^{\infty} N_{m-4,n} \bar{r}^m \right) = 0 \end{aligned}$$

From the above equation, one of the series begins with $m = 4$; therefore, let us separate and evaluate the first and

second series (2 terms) corresponding to $m=0,1,2,3$ and $m=2,3$, respectively:

$$\begin{aligned} & 0N_{0,n}\bar{r}^0 + 1N_{1,n}\bar{r}^1 + 4N_{2,n}\bar{r}^2 + 9N_{3,n}\bar{r}^3 + \\ & \sum_{m=4}^{\infty} m^2 N_{m,n} \bar{r}^m + \lambda_n^2 (N_{0,n} \bar{r}^2 + N_{1,n} \bar{r}^3) + \\ & \lambda_n^2 \sum_{m=4}^{\infty} (N_{m-2,n} - N_{m-4,n}) \bar{r}^m = 0 \\ & \Rightarrow 0N_{0,n} + 1N_{1,n} \bar{r}^1 + \\ & [4N_{2,n} + \lambda_n^2 N_{0,n}] \bar{r}^2 + [9N_{3,n} + \lambda_n^2 N_{1,n}] \bar{r}^3 + \\ & \sum_{m=4}^{\infty} [m^2 N_{m,n} + \lambda_n^2 (N_{m-2,n} - N_{m-4,n})] \bar{r}^m = 0 \end{aligned}$$

Because the sum of the terms is 0, power series must be equal to 0.

And by letting $N_{0,n} = 1$ (unity), we have the following:

1. $0N_{0,n} = 0 \Rightarrow 0 \times 1$, but $N_{0,n} = 1$ (assumption)
2. $1N_{1,n} = 0 \Rightarrow N_{1,n} = 0$
3. $4N_{2,n} + \lambda_n^2 N_{0,n} = 0 \Rightarrow 4N_{2,n} + \lambda_n^2 = 0$
 $\Rightarrow 4N_{2,n} = -\lambda_n^2$ since $N_{0,n} = 1 \Rightarrow N_{2,n} = -\lambda_n^2 / 4$
4. $9N_{3,n} + \lambda_n^2 N_{1,n} = 0 \Rightarrow 9N_{3,n} = 0$, since $N_{1,n} = 0$,
 from step 2 above $\Rightarrow N_{3,n} = 0$

From the above 4 steps, we can conclude that for any odd values of m , $N_{m,n} = 0$

Therefore, equation (29) which is $R_n(\bar{r}) = \sum_{m=0,2,\dots}^{\infty} N_{m,n} \bar{r}^m$ a solution, only if the summation is taken only for even values of m :

$$\begin{aligned} & 5. m^2 N_{m,n} + \lambda_n^2 (N_{m-2,n} - N_{m-4,n}) = 0 \\ & \Rightarrow N_{m,n} = -\frac{\lambda_n^2}{m^2} (N_{m-2,n} - N_{m-4,n}), \text{ for } m = 4, 6, 8, \dots \end{aligned} \quad (36)$$

Now using equation (36), we can find the values for $N_{4,n}, N_{6,n}, N_{8,n}, \dots$

But from the above, we have $N_{0,n} = 1$ and $N_{2,n} = -\lambda_n^2 / 4$.

Hence, for $m = 4$,

$$N_{4,n} = -\frac{\lambda_n^2}{4^2} (N_{2,n} - N_{0,n})$$

Then, substituting equation (4) for m , we will have the following:

$$\begin{aligned} N_{4,n} &= -\frac{\lambda_n^2}{16} (N_{2,n} - N_{0,n}) = -\frac{\lambda_n^2}{16} (-\lambda_n^2 / 4 - 1) \text{ since } N_{0,n} = 1 \\ &\Rightarrow N_{4,n} = \frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16} \end{aligned}$$

For $m = 6$,

$$\begin{aligned} N_{6,n} &= -\frac{\lambda_n^2}{36} (N_{4,n} - N_{2,n}) = -\frac{\lambda_n^2}{36} \left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16} - \left(-\frac{\lambda_n^2}{4} \right) \right) \\ &\Rightarrow N_{6,n} = -\frac{\lambda_n^6}{2304} - \frac{5\lambda_n^4}{576} \end{aligned}$$

For $m = 8$,

$$\begin{aligned} N_{8,n} &= -\frac{\lambda_n^2}{64} (N_{6,n} - N_{4,n}) = -\frac{\lambda_n^2}{64} \left(-\frac{\lambda_n^6}{2304} - \frac{5\lambda_n^4}{576} - \left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16} \right) \right) \\ &= -\frac{\lambda_n^2}{64} \left(-\frac{\lambda_n^6}{2304} - \frac{89\lambda_n^4}{36864} - \frac{\lambda_n^2}{16} \right) \\ &\Rightarrow N_{8,n} = \frac{\lambda_n^8}{147456} + \frac{14\lambda_n^6}{36864} + \frac{\lambda_n^4}{1024} \end{aligned}$$

For $m = 10$,

$$\begin{aligned} N_{10,n} &= -\frac{\lambda_n^2}{100} (N_{8,n} - N_{6,n}) \\ &= -\frac{\lambda_n^2}{100} \left(\frac{\lambda_n^8}{147456} + \frac{14\lambda_n^6}{36864} + \frac{\lambda_n^4}{1024} - \left(-\frac{\lambda_n^6}{2304} - \frac{5\lambda_n^4}{576} \right) \right) \\ &= -\frac{\lambda_n^2}{100} \left(\frac{\lambda_n^8}{147456} + \frac{14\lambda_n^6}{36864} + \frac{89\lambda_n^4}{9216} \right) \\ &\Rightarrow N_{10,n} = -\frac{\lambda_n^{10}}{14745600} - \frac{14\lambda_n^8}{3686400} - \frac{89\lambda_n^6}{921600} \end{aligned}$$

In a similar fashion, we can solve $N_{12,n}, N_{14,n}, \dots$

Therefore, equation (29), ie, $R_n(\bar{r}) = \sum_{m=0,2,4,\dots}^{\infty} N_{m,n} \bar{r}^m$, becomes

$$\begin{aligned} R_n(\bar{r}) &= (N_{0,n}) \bar{r}^0 + (N_{2,n}) \bar{r}^2 + (N_{4,n}) \bar{r}^4 + (N_{6,n}) \bar{r}^6 \\ &\quad + (N_{8,n}) \bar{r}^8 + (N_{10,n}) \bar{r}^{10} + \dots \end{aligned}$$

If we substitute the above values, we will have the following:

$$\begin{aligned} R_n(\bar{r}) &= 1 - \frac{\lambda_n^2}{4} \bar{r}^2 + \left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16} \right) \bar{r}^4 - \left(\frac{\lambda_n^6}{2304} + \frac{\lambda_n^4}{576} \right) \bar{r}^6 \\ &\quad + \left(\frac{\lambda_n^8}{147456} + \frac{14\lambda_n^6}{36864} + \frac{\lambda_n^4}{1024} \right) \bar{r}^8 \\ &\quad - \left(\frac{\lambda_n^{10}}{14745600} + \frac{14\lambda_n^8}{3686400} + \frac{89\lambda_n^6}{921600} \right) \bar{r}^{10} + \dots \\ R_n(\bar{r}) &= 1 - \frac{\lambda_n^2}{4} \bar{r}^2 + \left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16} \right) \bar{r}^4 + O(\bar{r}^6) \end{aligned} \quad (37)$$

Hence, by truncating the term with order 6, we have the following:

$$R_n(\bar{r}) = 1 - \frac{\lambda_n^2}{4} \bar{r}^2 + \left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16} \right) \bar{r}^4 \quad (38)$$

From the boundary condition equation (28), we can get the value of λ_n by substituting equation (29):

$$\frac{dR_n}{d\bar{r}} + Sh_w R_n = 0, \text{ at } \bar{r} = 1 \quad (39)$$

$$\frac{dR_n}{d\bar{r}} = -\frac{\lambda_n^2}{2}\bar{r} + \left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16}\right)\bar{r}^3 \quad (40)$$

Then, substituting equations (38) and (40) in equation (28), we get the following:

$$\begin{aligned} & -\frac{\lambda_n^2}{2}\bar{r} + \left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16}\right)\bar{r}^3 + Sh_w \left(1 - \frac{\lambda_n^2}{4}\bar{r}^2 + \left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16}\right)\bar{r}^4\right) \\ & = 0 \text{ at } \bar{r} = 1 \\ & -\frac{\lambda_n^2}{2} + \left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16}\right) + Sh_w \left(1 - \frac{\lambda_n^2}{4} + \left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16}\right)\right) \\ & = 0 \text{ (Substituting 1 for } \bar{r}) \\ & \Rightarrow (1 + Sh_w)\lambda_n^4 - (28 + 12Sh_w)\lambda_n^2 + 64Sh_w = 0 \end{aligned}$$

The above equation is of the quadratic form.

Now let $\lambda_n^2 = k$, then substitution leads to the following:

$$(1 + Sh_w)k^2 - (28 + 12Sh_w)k + 64Sh_w = 0$$

Then, using the general quadratic formula,

$$\begin{aligned} k = \lambda_n^2 &= \frac{(28 + 12Sh_w) \pm \sqrt{(28 + 12Sh_w)^2 - 4(1 + Sh_w)(28 + 12Sh_w)}}{2(1 + Sh_w)} \\ &\Rightarrow \frac{(28 + 12Sh_w) \pm \sqrt{96Sh_w^2 + 512Sh_w + 672}}{2(1 + Sh_w)} \\ &\Rightarrow \frac{(14 + 6Sh_w) \pm \sqrt{24Sh_w^2 + 128Sh_w + 168}}{(1 + Sh_w)} \end{aligned}$$

Hence, from the above, we can obtain 4 values of λ_n such as $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$, ie,

But, we suppose that $\lambda_n^2 = k$:

$$\Rightarrow \lambda_n = \pm\sqrt{k}$$

Therefore,

$$\begin{aligned} \lambda_0 &= \sqrt{\frac{(14 + 6Sh_w) + \sqrt{24Sh_w^2 + 128Sh_w + 168}}{(1 + Sh_w)}} \\ \lambda_1 &= -\sqrt{\frac{(14 + 6Sh_w) + \sqrt{24Sh_w^2 + 128Sh_w + 168}}{(1 + Sh_w)}} \\ \lambda_2 &= \sqrt{\frac{(14 + 6Sh_w) - \sqrt{24Sh_w^2 + 128Sh_w + 168}}{(1 + Sh_w)}} \\ \lambda_3 &= -\sqrt{\frac{(14 + 6Sh_w) - \sqrt{24Sh_w^2 + 128Sh_w + 168}}{(1 + Sh_w)}} \end{aligned}$$

From equation (25), we have the following:

$$\begin{aligned} R_n(\bar{r}) &= 1 - \frac{\lambda_n^2}{4}\bar{r}^2 + \left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16}\right)\bar{r}^4 \\ \Rightarrow R_0(\bar{r}) &= 1 - \frac{\lambda_0^2}{4}\bar{r}^2 + \left(\frac{\lambda_0^4}{64} + \frac{\lambda_0^2}{16}\right)\bar{r}^4 \\ \Rightarrow R_1(\bar{r}) &= 1 - \frac{\lambda_1^2}{4}\bar{r}^2 + \left(\frac{\lambda_1^4}{64} + \frac{\lambda_1^2}{16}\right)\bar{r}^4 \\ \Rightarrow R_2(\bar{r}) &= 1 - \frac{\lambda_2^2}{4}\bar{r}^2 + \left(\frac{\lambda_2^4}{64} + \frac{\lambda_2^2}{16}\right)\bar{r}^4 \\ \Rightarrow R_3(\bar{r}) &= 1 - \frac{\lambda_3^2}{4}\bar{r}^2 + \left(\frac{\lambda_3^4}{64} + \frac{\lambda_3^2}{16}\right)\bar{r}^4 \end{aligned}$$

Using the condition for the orthogonality of R_n' s on the interval $[0, 1]$ and equation (26), we can find the values of A_n' s:

$$\begin{aligned} A_n &= \frac{\int_0^1 R_n(\bar{r}) d\bar{r}}{\int_0^1 (R_n(\bar{r}) d\bar{r})^2 d\bar{r}} \\ &= \frac{\int_0^1 \left(1 - \frac{\lambda_n^2}{4}\bar{r}^2 + \left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16}\right)\bar{r}^4\right) d\bar{r}}{\int_0^1 \left(1 - \frac{\lambda_n^2}{4}\bar{r}^2 + \left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16}\right)\bar{r}^4\right)^2 d\bar{r}} \\ &= \frac{\int_0^1 \left(1 - \frac{\lambda_n^2}{4}\bar{r}^2 + \left(\frac{3\lambda_n^4}{32} + \frac{\lambda_n^2}{8}\right)\bar{r}^4 - \frac{\lambda_n^2}{2}\left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16}\right)\bar{r}^6 + \left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16}\right)^2\bar{r}^8\right) d\bar{r}}{\int_0^1 \left(1 - \frac{\lambda_n^2}{12} + \frac{1}{5}\left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16}\right)\right) d\bar{r}} \\ \Rightarrow A_n &= \frac{1 - \frac{\lambda_n^2}{6} + \frac{1}{5}\left(\frac{3\lambda_n^4}{32} + \frac{\lambda_n^2}{8}\right) - \frac{\lambda_n^2}{14}\left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16}\right) + \frac{1}{\square}\left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16}\right)^2}{1 - \frac{\lambda_n^2}{12} + \frac{1}{5}\left(\frac{\lambda_n^4}{64} + \frac{\lambda_n^2}{16}\right)} \end{aligned}$$

Then, substituting the values of A_n and R_n into equation (24), we get the following:

$$\begin{aligned} \bar{c} &= A_0 R_0 e^{(-\lambda_0^2 \bar{z})} + A_1 R_1 e^{(-\lambda_1^2 \bar{z})} + A_2 R_2 e^{(-\lambda_2^2 \bar{z})} \\ &\quad + A_3 R_3 e^{(-\lambda_3^2 \bar{z})} + A_4 R_4 e^{(-\lambda_4^2 \bar{z})} \end{aligned}$$

Which is the solution for

$$\bar{c} = \sum_{n=0}^3 A_n R_n e^{(-\lambda_n^2 \bar{z})}$$

Discussion and Conclusions

As it can be seen clearly in Figure 1, of the "Introduction" section, when the length of the dialyzer increases, the solutes in the blood which comes from the patient's body will have more

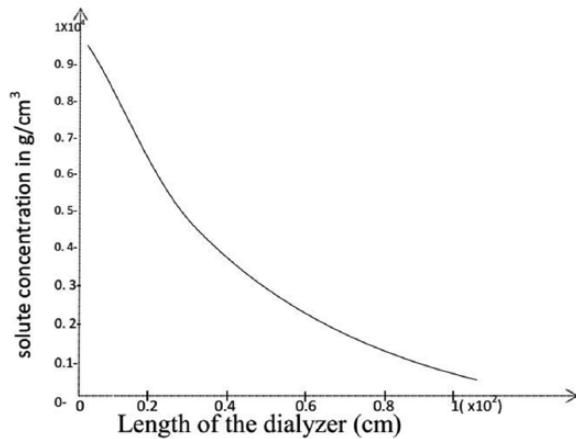


Figure 3. The graph of length of the dialyzer versus the concentration solute (urea).

possibility or places to diffuse to the dialysate through the membrane of the dialyzer. So the concentration of urea (solute) in the blood will greatly decrease when the length of the dialyzer increases. In general, the clearance of the dialyzer depends on the length of the dialyzer, ie, the tube dialyzer is best or effective when it is longer. This relationship is shown in the graph in Figure 3.

The amount of solutes in the blood is also greatly affected by the surface area of the membrane and its thickness. That is, when the surface area of the membrane increases, more

solutes from the blood will diffuse to the dialysate. Hence, the concentration of urea in the blood will greatly decrease. But when its thickness increases, the solutes will need a force to pass the membrane and diffuse with the dialysate. Therefore, the amount of solutes in the blood will never decrease as desired.

Therefore, the concluding remarks are as follows:

- Longer dialyzers are more efficient than shorter dialyzers.
- Dialyzers with thin membranes are more efficient than dialyzers with thick membranes.
- Dialyzers with more membrane surface area are more efficient than dialyzers with low membrane surface area.

Author Contributions

YN explains in this article the biological functions of kidneys. YN wrote the first draft of the manuscript and the mathematical formulations related to a model of Tube Dialyzer.

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